Variational bounds on energy dissipation in incompressible flows. II. Channel flow

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A variational principle for lower bounds on the time-averaged mass flux for Newtonian fluids driven by a pressure gradient in a channel is derived from the incompressible Navier-Stokes equations. When supplied with appropriate test background flow fields, the variational formulation produces explicit estimates for the friction coefficient. These rigorous bounds are compared with the predictions of conventional turbulence theory.

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I. INTRODUCTION

Most theoretical approaches to turbulence consist of approximate treatments ranging from the imposition of statistical assumptions and moment hierarchy truncations, to the introduction of scaling hypotheses [1]. In this paper we focus on a specific fundamental problem, the rate of mass transport in a pressure-gradient-driven channel flow, with the goal of deriving quantitative results directly from the equations of motion. We establish a practical framework for the rigorous estimation of the viscous energy dissipation rate, and thus the mass flux, directly from the incompressible Navier-Stokes equations for a Newtonian fluid. Our approach is to derive a variational principle for lower bounds on the time-averaged mass transport rate, utilizing a decomposition that we refer to as the background flow method. The variational principle applies to both laminar and turbulent flows, and leads to rigorous predictions free from secondary hypotheses or uncontrolled approximations.

This paper is the second in a series of three that follow up an earlier short presentation of some of these results [2]. In the previous paper we applied the same general approach to a boundary-driven shear layer [3], while the third paper in the series deals with the problem of thermal convection where bounds on the energy dissipation rate lead to estimates for the rate of convective heat transport [4]. The application developed here is to a flow driven by a pressure gradient, where the averaged energy dissipation rate is directly related to the global rate of mass transport, and to the effective friction coefficient.

Suppose an incompressible Newtonian fluid is confined to a rectangular channel as illustrated in Fig. 1. A uniform pressure gradient of magnitude P/L_x in the x direction drives the flow. Without loss of generality mass units can be chosen so that the density $\rho=1$, and we denote the kinematic viscosity by ν. The fluid's velocity vector field $\mathbf{u}(\mathbf{x},t)=(u_1,u_2,u_3)$ satisfies the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=\nu\Delta\mathbf{u}+\mathbf{i}\frac{P}{L_x}\ , \tag{1.1}$$

$$\nabla\cdot\mathbf{u}=0\ , \tag{1.2}$$

where $p(\mathbf{x},t)$ is the pressure field determined by the divergence-free condition on $\mathbf{u}$, and $\mathbf{i}$ is the unit vector in the x direction (1 direction). The boundary conditions on $\mathbf{u}$ are taken to be no-slip on the planes $z=0$ and h, and periodic in the x and y directions—with periods L_x and L_y, respectively—for both $\mathbf{u}$ and p. The initial velocity vector field $\mathbf{u}(\mathbf{x},0)=\mathbf{u}_0(\mathbf{x})$ is square integrable.

The instantaneous mass flux in the x direction is

$$\Phi(t)=\int_0^{L_y}dy\int_0^{h}dz\ u_1(x,y,z,t)\ . \tag{1.3}$$

In light of the incompressibility condition and the boundary conditions, this flux is independent of x. The instantaneous energy dissipation rate (per unit density) in the fluid is

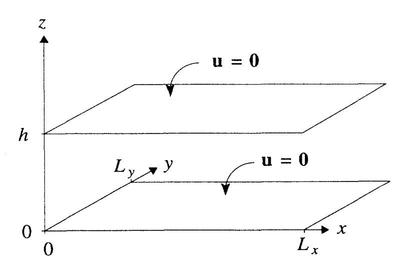

FIG. 1. Fluid is confined between parallel plates of dimension $L_x\times L_y$, separated by a gap h in the z direction. Boundary conditions are $\mathbf{u}=0$ for $z=0$ and $z=h$, and periodic in the x and y directions. The flow is maintained by a pressure gradient P/L_x in the x direction.

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$$\nu\|\nabla\mathbf{u}\|_2^2=\nu\sum_{i,j=1}^{d}\left\|\frac{\partial u_i}{\partial x_j}\right\|_2^2 , \tag{1.4}$$

where $\|f\|_2$ denotes the L_2 norm of a function $f(\mathbf{x})$:

$$\|f\|_2=\left[\int_\Omega |f(\mathbf{x})|^2 d\mathbf{x}\right]^{1/2} . \tag{1.5}$$

We are concerned with time-averaged fluxes

$$\langle\Phi\rangle_T=\frac{1}{T}\int_0^T\Phi(t)dt \tag{1.6}$$

and time-averaged dissipation rates

$$\langle\nu\|\nabla\mathbf{u}\|_2^2\rangle_T=\frac{1}{T}\int_0^T\nu\|\nabla\mathbf{u}(\cdot,t)\|_2^2dt . \tag{1.7}$$

The connection between the mass flux and the energy dissipation is seen in the energy evolution equation derived from the Navier-Stokes equations (1.1) by dotting with $\mathbf{u}$, integrating over space, and integrating by parts using the boundary conditions:

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{u}\|_2^2+\nu\|\nabla\mathbf{u}\|_2^2=P\Phi . \tag{1.8}$$

The kinetic energy is constant on average in a steady state, so the average rate of viscous energy dissipation per unit applied pressure is precisely the time-averaged mass flux. Estimates of the average energy dissipation rate and bounds on the average flux are thus interchangeable, and in this application we will focus directly on the mass flux.

The rest of this paper is organized as follows. In Sec. II we formulate a variational principle for lower bounds on the time-averaged mass flux for flow in a rectangular channel driven by a pressure gradient. The basis of the principle is a decomposition of the flow field into a background and a fluctuation reminiscent of, but distinct from, the Reynolds decomposition into mean and fluctuating components. In Sec. III we use elementary functional estimates to produce appropriate trial background flows, resulting in explicit estimates. In Sec. IV we compare these rigorous bounds with the predictions of conventional turbulence theory. This comparison leads naturally to questions of how the estimates might be improved, and we present some possible approaches in that direction.

II. VARIATIONAL PRINCIPLE

Long-time limits of the finite-time averages need not exist, even if finite-time averages are bounded. Moreover, the long-time averages in Eqs. (1.6) and (1.7) need not be unique, for even if the limit $T\to\infty$ did exist, it would generally depend on the initial conditions. Eventual bounds on the long-time averages nevertheless exist.

The variational principle for lower bounds on the smallest possible time-averaged mass flux expresses the lower estimate as a supremum over a set of functions $U(z)$:

Theorem. For every solution of Eqs. (1.1) and (1.2), with the prescribed boundary conditions,

$$\liminf_{T\to\infty}\langle\Phi\rangle_T\ge\sup\left\{\frac{1}{12}\frac{Ph^3L_y}{\nu L_x}-\frac{\nu L_xL_y}{P}\int_0^h\left[U'(z)-\frac{P}{2\nu L_x}(h-2z)\right]^2dz\,\middle|\,U(0)=0=U(h),H_U\ge0\right\} , \tag{2.1}$$

where U' is the derivative of $U(z)$ and H_U is a quadratic functional that depends parametrically on $U(z)$,

$$H_U\{\mathbf{v}\}=\int_0^{L_x}dx\int_0^{L_y}dy\int_0^h dz\left\{\frac{\nu}{2}|\nabla\mathbf{v}|^2+v_1v_3U'(z)\right\} , \tag{2.2}$$

defined for divergence-free vector fields $\mathbf{v}(\mathbf{x})$ satisfying $\mathbf{u}$'s boundary conditions.

Proof. Let $\mathbf{u}(\mathbf{x},t)$ be a solution of the Navier-Stokes equations starting from initial square-integrable divergence-free velocity vector field $\mathbf{u}(\mathbf{x},0)$. The integrated form of the energy evolution equation for $\mathbf{u}$ (integrated over time from 0 to T) is

$$\frac{1}{2T}\|\mathbf{u}(\cdot,T)\|_2^2+\langle\nu\|\nabla\mathbf{u}\|_2^2\rangle_T=\frac{1}{2T}\|\mathbf{u}(\cdot,0)\|_2^2+P\langle\Phi\rangle_T . \tag{2.3}$$

Decompose $\mathbf{u}$ according to

$$\mathbf{u}(\mathbf{x},t)=\mathbf{i}U(z)+\mathbf{v}(\mathbf{x},t) , \tag{2.4}$$

where $\mathbf{v}$ satisfies the same boundary conditions as $\mathbf{u}$ and U vanishes at $z=0$ and h. We refer to the stationary component of Eq. (2.4), namely $\mathbf{i}U$, as the background flow, and to $\mathbf{v}$ as the fluctuation. Writing down the evolution equation for $\mathbf{v}$ and performing operations similar to those that lead to Eq. (2.3), we find

$$\frac{1}{T}\frac{1}{2}\|\mathbf{v}(\cdot,T)\|_2^2+\langle\nu\|\nabla\mathbf{v}\|_2^2\rangle_T+\left\langle\int\nu U'\frac{\partial v_1}{\partial z}d\mathbf{x}\right\rangle_T+\left\langle\int U'v_1v_3d\mathbf{x}\right\rangle_T=\frac{1}{T}\frac{1}{2}\|\mathbf{v}_0\|_2^2+\frac{P}{L_x}\left\langle\int v_1d\mathbf{x}\right\rangle_T . \tag{2.5}$$

Use the fact that

$$\Phi=L_y\int_0^h dz\,U(z)+\int_0^{L_y}dy\int_0^h dz\,v_1 \tag{2.6}$$

and

$$\|\nabla\mathbf{u}\|_2^2=L_xL_y\int_0^h dz\,U'(z)^2+2\int_0^{L_x}dx\int_0^{L_y}dy\int_0^h dz\,U'(z)\frac{\partial v_1}{\partial z}+\|\nabla\mathbf{v}\|_2^2 , \tag{2.7}$$

along with Eq. (2.3), to deduce from Eq. (2.5) that

$$\langle \Phi \rangle_T = 2L_y \int_0^h dz\, U(z) - \frac{\nu L_x L_y}{P} \int_0^h dz\, U'(z)^2 + \frac{2}{P}\left\langle \int \left\{ \frac{\nu}{2}|\nabla \mathbf{v}|^2 + U'(z) v_1 v_3 \right\} d\mathbf{x} \right\rangle_T + \frac{1}{PT}\|\mathbf{v}(\cdot,T)\|_2^2 + \frac{1}{2PT}\|\mathbf{u}(\cdot,0)\|_2^2 - \frac{1}{PT}\|\mathbf{v}(\cdot,0)\|_2^2 - \frac{1}{2PT}\|\mathbf{u}(\cdot,T)\|_2^2 . \quad (2.8)$$

It can be shown by standard methods (see, for example, the analysis in Ref. [3]) that $\|\mathbf{u}(\cdot,t)\|_2$ and $\|\mathbf{v}(\cdot,t)\|_2$ remain *a priori* bounded uniformly in time, implying that the terms with coefficients $1/T$ in Eq. (2.8) will vanish as $T \to \infty$. Then invoking the positivity of H_U, an eventual lower bound for $\langle \Phi \rangle_T$ is given by the first two terms on the right hand side of Eq. (2.8). The assertion follows immediately from a completion of the square. ■

We will refer to the positivity condition on H_U as the spectral constraint on the trial background flow profile $U(z)$. This is because the non-negativity of H_U is equivalent to the non-negativity of the spectrum of the associated self-adjoint eigenvalue problem

$$\begin{aligned} \lambda v_1 &= -\nu \Delta v_1 + \frac{\partial p}{\partial x} + U'(z) v_3 , \\ \lambda v_2 &= -\nu \Delta v_2 + \frac{\partial p}{\partial y} , \\ \lambda v_3 &= -\nu \Delta v_3 + \frac{\partial p}{\partial z} + U'(z) v_1 , \\ 0 &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \end{aligned} \quad (2.9)$$

for vector fields $\mathbf{v}$ vanishing at $z=0$ and h and periodic in the x and y directions. The spectral constraint ($\lambda_{\min} \geq 0$) makes the variational problem for the extremum flow profile somewhat unusual, and one must use care in the application of standard constrained variational calculus to derive the Euler-Lagrange equations for the optimal background profile $U(z)$ [5]. We will return to this issue in Sec. IV, where we indicate how the Euler-Lagrange equations are derived. Optimization notwithstanding, this variational formulation will be useful for establishing rigorous bounds: to derive a lower estimate on the time-averaged mass flux we need only produce a trial background flow profile $U(z)$ satisfying the boundary conditions and the spectral constraint. In Sec. III we construct appropriate flows, and show how the spectral constraint can be verified by elementary functional methods.

III. EXPLICIT BOUNDS

There is an exact laminar stationary solution to the Navier-Stokes equations in this geometry, known as the plane Poiseuille flow:

$$\mathbf{u} = \mathbf{i}\frac{P}{2\nu L_x} z(h-z) , \quad p = \text{const} . \quad (3.1)$$

To see how this arises in the framework of the variational principle, ignore the spectral constraint and note that the supremum of the lower estimates in Eq. (2.1) is achieved when the argument of the integral vanishes, i.e., when

$$U'(z) = \frac{P}{2\nu L_x}(h-2z) , \quad (3.2)$$

with boundary conditions $U(0)=0=U(h)$. Integrating, we see that the solution is precisely the Poiseuille profile in Eq. (3.1), sketched in Fig. 2.

The mass flux in this laminar flow state is

$$\Phi^{\text{Poiseuille}} = \frac{1}{12}\frac{Ph^3 L_y}{\nu L_x} = \overline{U}^{\text{Poiseuille}} L_y h , \quad (3.3)$$

where we introduce the average speed defined by the flux

$$\overline{U} = \frac{\Phi}{L_y h} , \quad (3.4)$$

which for the Poiseuille profile is

$$\overline{U}^{\text{Poiseuille}} = \frac{1}{12}\frac{Ph^2}{\nu L_x} . \quad (3.5)$$

In general a Reynolds number may be defined by this velocity scale along with the channel width and the viscosity,

$$R = \frac{\overline{U}h}{\nu} . \quad (3.6)$$

The friction coefficient C_f, defined according to

$$C_f \equiv \frac{Ph}{L_x \overline{U}^2} , \quad (3.7)$$

provides a dimensionless ratio of the applied pressure gradient to the square of the flow velocity scale. For the Poiseuille flow the friction coefficient is

$$C_f^{\text{Poiseuille}} = \frac{12}{R} . \quad (3.8)$$

When the applied pressure is low enough, the flux in the Poiseuille profile is a lower bound on all long-time averaged fluxes, and $C_f^{\text{Poiseuille}}$ is an upper bound on the friction coefficient. From the point of view of the variational principle, this is the case when the Poiseuille flow profile satisfies the spectral constraint. In fact, more is true. The nonlinear stability of the planar Poiseuille flow is ensured by the positivity of the functional

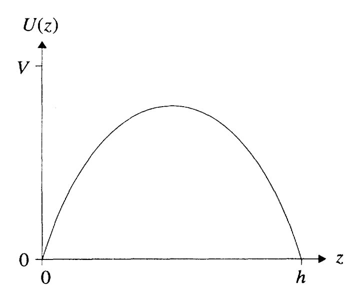

FIG. 2. The Poiseuille flow profile.

$$I\{\mathbf{v}\}=\int_0^{L_x}dx\int_0^{L_y}dy\int_0^{h}dz\left\{\nu|\nabla\mathbf{v}|^2+\frac{P}{2\nu L_x}(h-2z)v_1v_3\right\}, \tag{3.9}$$

defined for divergence-free vector fields $\mathbf{v}(\mathbf{x})$ satisfying $\mathbf{u}$'s boundary conditions [6]. (This functional is almost the same as the functional H_U entering into the spectral constraint, for U replaced by the Poiseuille flow profile. The difference is that the viscosity in I is replaced by half the viscosity in H_U.) The positivity of I defines a critical pressure P_c, below which plane the Poiseuille flow is the unique asymptotic state of the system, where the unique long-time averaged mass flux is given in Eq. (3.3). For $P>P_c$ this argument no longer guarantees that the flux in Eq. (3.3) is the minimum possible long-time average, and we must appeal to the variational principle.

In order to derive bounds from the variational principle for pressure gradients higher than P_c/L_x, it is necessary to produce a trial background flow profile $U(z)$ which vanishes at $z=0$ and h, and satisfies the spectral constraint. What works is a two-parameter profile of the form

$$U(z)=\begin{cases}\frac{V}{\delta}z\ , & 0\le z\le\delta\\ V\ , & \delta\le z\le h-\delta\\ \frac{V}{\delta}(h-z)\ , & h-\delta\le z\le h\ ,\end{cases} \tag{3.10}$$

sketched in Fig. 3. As will be shown below, V and δ may be chosen so that the spectral constraint is satisfied.

Rather than solve the eigenvalue problem in Eq. (2.9) in order to check the spectral constraint, we may perform the analysis directly in terms of the quadratic form. For the profile in Eq. (3.10), the second term in H_U is estimated in terms of the first term by standard methods as follows (for details, see the analogous calculation in Ref. [3]):

$$\frac{V}{\delta}\left|\int_0^{L_x}dx\int_0^{L_y}dy\int_0^{\delta}dz\,v_1v_3-\int_0^{L_x}dx\int_0^{L_y}dy\int_{h-\delta}^{h}dz\,v_1v_3\right|\le\frac{V\delta}{4\sqrt{2}}\|\nabla\mathbf{v}\|_2^2\ . \tag{3.11}$$

Hence H_U is bounded from below according to

$$H_U\{\mathbf{v}\}\ge\left[\frac{\nu}{2}-\frac{V\delta}{4\sqrt{2}}\right]\|\nabla\mathbf{v}\|_2^2\ , \tag{3.12}$$

and non-negativity is ensured by the choice

$$\delta=2\sqrt{2}\frac{\nu}{V}\ . \tag{3.13}$$

For the flow profile in Eq. (3.10) subject to (3.13),

$$\liminf_{T\to\infty}\langle\Phi\rangle_T\ge 2L_y\int_0^h U(z)dz-\frac{\nu L_xL_y}{P}\int_0^h U'(z)^2dz$$
$$=2L_yhV-4\sqrt{2}L_y\nu-\frac{L_xL_y}{P\sqrt{2}}V^3\ . \tag{3.14}$$

Maximizing this lower bound with the choice

$$V^2=\frac{2\sqrt{2}}{3}\frac{Ph}{L_x}\ , \tag{3.15}$$

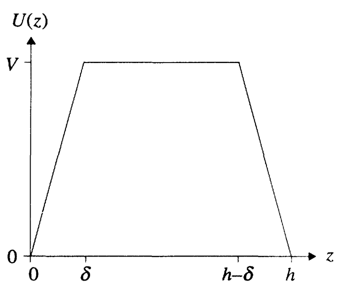

FIG. 3. Trial background flow profile $U(z)$.

we conclude that

$$\liminf_{T\to\infty}\langle\Phi\rangle_T\ge\left[\frac{32\sqrt{2}}{27}\frac{L_y^2h^3P}{L_x}\right]^{1/2}-4\sqrt{2}L_y\nu\ . \tag{3.16}$$

This is the essence of our result. Comparing with Eq. (3.3) we see that while the mass flux is directly proportional to the pressure drop in the laminar state, we can only be sure (based on analysis of the Navier-Stokes equations) that its average is at least proportional to the square root of the pressure drop for strongly driven—including turbulent—flows.

This result may be reexpressed in terms of a friction coefficient and a Reynolds number by defining the minimum velocity scale as in Eq. (3.4),

$$\overline{U}=\frac{1}{L_yh}\liminf_{T\to\infty}\langle\Phi\rangle_T\ , \tag{3.17}$$

and using it to define the Reynolds number as in Eq. (3.6), and the friction coefficient as in Eq. (3.7). Then our upper bound may be cast in a dimensionless form [analogous to the laminar case in Eq. (3.8)]

$$C_f\le\frac{27}{32}\frac{1}{\sqrt{2}}\left[1+\frac{4\sqrt{2}}{R}\right]^2\ . \tag{3.18}$$

IV. DISCUSSION

We have derived a rigorous upper bound on the friction coefficient, valid when $P\ge(48/\sqrt{2})\nu^2L_x/h^3$, so that the boundary layer thickness $\delta\le h/2$. Asymptotically at high Reynolds numbers this bound takes the form

$$C_f \le \text{const} \ , \tag{4.1}$$

which is independent of the viscosity ν, the fundamental local friction coefficient. This result, which may at first seem surprising, is in accord with Kolmogorov's scaling hypothesis for turbulence [7]. That theory asserts that the rate of energy dissipation in fully developed ($R \to \infty$) turbulence ought not depend on the viscosity. The energy dissipation is directly proportional to the mass flux for the problem under consideration, and this translates directly into the viscosity independence of the global effective friction coefficient.

Kolmogorov's hypothesis—usually invoked for homogeneous isotropic geometries—is known from experiment to require corrections in the presence of boundaries, and the problem of wall-bounded turbulence is necessarily more complicated. Classical turbulence theory in the form of Prandtl's mixing length closure approximation [8] (see also Refs. [1] or [3] for a simple version of the closure) predicts the so-called logarithmic friction law at high Reynolds numbers:

$$C_f \sim \frac{1}{(\ln R)^2} \ . \tag{4.2}$$

This approximate theory, which with one or two adjustable parameters provides a very good fit to existing experimental data for high Reynolds number pipe flow, predicts a relatively weak logarithmic dependence on the viscosity. So the rigorous upper estimate in Eq. (4.1)—as far as the Reynolds number dependence is concerned—appears to be sharp to within logarithms. Such results are encouraging, but it is natural to ask where there might be room for improvement in the estimates.

Two short cuts were taken in Sec. III in order to obtain explicit rigorous results. First, primarily for analytical convenience, trial profiles were sampled from a very restricted class of functions, namely simple piecewise linear functions as in Fig. 3. Second, elementary and somewhat crude estimates were employed to ensure that the spectral constraint was satisfied rather than verifying it explicitly. The constraint is oversatisfied, resulting in an overestimate of the best bound on C_f. What is neglected is the divergence-free restriction on the functions in the domain of the functional in Eq. (2.2). To optimally check the spectral constraint for a given test background profile, the eigenvalue problem in Eq. (2.9) should be solved, and the lowest eigenvalue determined as a functional of the test background profile. In the case of piecewise linear $U(z)$, the eigenvalue problem is a set of linear, piecewise-constant coefficient differential equations. For the shear flow case (Refs. [2] and [3]) the constraint has been applied exactly for the piecewise linear profiles [9], yielding an order of magnitude improvement in the prefactor, but the same Reynolds number dependence for the frictional drag. Hence it is apparent that even with the spectral constraint precisely enforced, the simple form of the trial background flow profile that we used will not lead to qualitative improvement in the upper bound on C_f.

Ideally the variational problem for the optimal background profile will be solved exactly to yield the best estimates that this method has to offer. There is hope that the extremum background profile for the variational problem may be solved: the first step in this direction is to derive the associated Euler-Lagrange equations. Of course the spectral constraint is not standard, but it can be handled as follows.

Transform the function to be varied, $U(z)$, to a variable function $W(z)$ according to

$$W(z) = U'(z) - \frac{P}{2\nu L_x}(h-2z) \ , \tag{4.3}$$

and rewrite the variational problem as

$$\Phi^{\text{Poiseuille}} - \liminf_{T\to\infty} \langle \Phi \rangle_T \le \inf \left\{ \frac{\nu L_x L_y}{P} \int_0^h W(z)^2 dz \,\middle|\, 0 = \int_0^h W(z) dz, \ H_U \ge 0 \right\} . \tag{4.4}$$

In terms of $W(z)$, the spectral constraint $H_U \ge 0$ is equivalently

$$0 \le \int_0^{L_x} dx \int_0^{L_y} dy \int_0^h dz \left\{ \frac{\nu}{2} |\nabla \mathbf{v}|^2 + \frac{P}{2\nu L_x}(h-2z) v_1 v_3 + W(z) v_1 v_3 \right\} , \tag{4.5}$$

or $\lambda_0 \ge 0$, where λ_0 is the lowest eigenvalue of the self-adjoint operator in

$$\begin{aligned} \lambda v_1 &= -\nu \Delta v_1 + \frac{\partial p}{\partial x} + \frac{P}{2\nu L_x}(h-2z) v_3 + W(z) v_3 \ , \\ \lambda v_2 &= -\nu \Delta v_2 + \frac{\partial p}{\partial y} \ , \\ \lambda v_3 &= -\nu \Delta v_3 + \frac{\partial p}{\partial z} + \frac{P}{2\nu L_x}(h-2z) v_1 + W(z) v_1 \ , \\ 0 &= \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z} \ . \end{aligned} \tag{4.6}$$

In the form of Eq. (4.4) it is clear that the best possible bound to be hoped for (the Poiseuille limit) is realized when $W \equiv 0$. But this is acceptable only if P is low enough for the spectral constraint to hold, a sufficient condition for which is the nonlinear stability of the Poiseuille flow.

Now observe that in the space of square-integrable functions with mean zero, the set of functions $W(z)$ satisfying the spectral constraint is convex. By this we mean that if $W_1(z)$ and $W_2(z)$ both satisfy the constraint, then for each $0 \le t \le 1$ the convex combination $tW_1 + (1-t)W_2$ also satisfies the constraint. This is most easily seen by noting that W appears linearly in H_U, so the inequality is preserved by replacing either W_1 or W_2 with a convex combination. What this means is that if the set of W's satisfying the spectral constraint does not include the origin of the function space, then the W in the set minimizing the distance to the origin must lie on the boundary of the set. Analytically this means that the constraint $\lambda_0 \geq 0$ can be replaced with $\lambda_0 = 0$. Because $\lambda_0 = \lambda_0[W]$ is a functional of W, albeit a complicated one, the spectral constraint is now expressed in a form suitable for the usual constrained variational calculus.

Define the functional $F[W]$ by

$$F[W] = \frac{1}{2}\int_0^h W(z)^2 dz - \alpha \int_0^h W(z) dz + \beta\lambda_0[W] , \tag{4.7}$$

where α and β are Lagrange multipliers. Then the Euler-Lagrange equation for the constrained variational problem is

$$0 = \frac{\delta F}{\delta W} = W(z) - \alpha + \beta \frac{\delta \lambda_0}{\delta W} , \tag{4.8}$$

where the variation of the ground state eigenvalue λ_0 with respect to the potential W is straightforward to compute via regular first order perturbation theory.

For simplicity let us specialize to the two dimensional channel flow problem ($v_2 = 0 = u_2$ and $\partial/\partial y = 0$). Presuming an e^{ikx} dependence for v_1, v_3, and p, and using the shorthand $u(z) = e^{-ikx} v_1$ and $w(z) = e^{-ikx} v_3$, the eigenvalue problem is

$$\lambda u = \nu \left[k^2 - \frac{d^2}{dz^2} \right] u + ikp + \frac{P}{2\nu L_x}(h - 2z) w + W(z) w ,$$

$$\lambda w = \nu \left[k^2 - \frac{d^2}{dz^2} \right] w + \frac{\partial p}{\partial z} + \frac{P}{2\nu L_x}(h - 2z) u + W(z) u , \tag{4.9}$$

$$0 = iku + \frac{\partial w}{\partial z} ,$$

The variation of the eigenvalue is

$$\frac{\delta \lambda}{\delta W} = ik \frac{w^* \frac{dw}{dz} - w \frac{dw^*}{dz}}{\int_0^h \left[\left| \frac{dw(z')}{dz'} \right|^2 + k^2 |w(z')|^2 \right] dz'} . \tag{4.10}$$

The Euler-Lagrange equation for the extremal profile, Eq. (4.8), expresses the optimal W in terms of the set (u, w, p) associated with $\lambda = 0$ in Eq. (4.9):

$$W(z) = \alpha - \beta ik \frac{w^* \frac{dw}{dz} - w \frac{dw^*}{dz}}{\int_0^h \left[\left| \frac{dw(z')}{dz'} \right|^2 + k^2 |w(z')|^2 \right] dz'} . \tag{4.11}$$

The mean-zero constraint on W determines one of the Lagrange multipliers, so that

$$W(z) = \alpha \left[1 - \frac{w^*(z) \frac{dw(z)}{dz} - w(z) \frac{dw^*(z)}{dz}}{\frac{1}{h} \int_0^h \left[w^*(z') \frac{dw(z')}{dz'} - w(z') \frac{dw^*(z')}{dz'} \right] dz'} \right] . \tag{4.12}$$

Inserting this into the equation for w with $\lambda = 0$ leads to a nonlinear boundary value problem in which α is to be adjusted so that a solution exists.

While the Euler-Lagrange equation has not been solved for this incompressible flow problem, a simpler variational problem with a spectral constraint has been solved exactly in Ref. [5], showing that the procedure is consistent and adding to the hope that progress in this direction will be possible. It is likely that a combination of numerical and asymptotic methods can be fruitfully brought to bear to solve this nonlinear problem.

Once the optimal profile has been obtained, the self-adjoint eigenvalue problem associated with the spectral constraint leads to consideration of the complete set of eigenfunctions. When the background profile is optimal, these eigenfunctions provide a basis that is adapted to turbulent flow problems, uniquely generated in an interesting way from the fundamental equations of motion. It will be interesting to look at the structure of these flow fields with the hope that elements of the turbulent dynamics may be illuminated in these coordinates (see Ref. [9] for further development of this idea applied to the piecewise linear background profile in the shear-flow problem). Using these adapted bases for Galerkin truncations, leading to finite dimensional dynamical systems models, is also an interesting open area for investigation.

A different variational approach to bounds on flow quantities was previously developed starting from the Reynolds decomposition into mean and fluctuation flows [10]. The predictions of that method, both the Reynolds number scaling and the magnitudes of prefactors, are generally the same as those derived in this paper directly from the Navier-Stokes equations but without the additional assumption that time averages exist, or that spatial averages are time independent.

In conclusion, the methods and results presented here illustrate the potential and utility of rigorous studies of the Navier-Stokes equations in problems of direct relevance to high Reynolds number turbulence. Although many mathematical challenges remain, the outlook is good for continued development of these techniques, resulting in improved analytical estimates and more fundamental physical understanding of the structure and behavior of fluid dynamical systems.

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